

# ON ESTIMATES FOR FULLY NONLINEAR PARABOLIC EQUATIONS ON RIEMANNIAN MANIFOLDS

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**ABSTRACT.** In this paper we present some new ideas to derive *a priori* second order estimates for a wide class of fully nonlinear parabolic equations. Our methods, which produce new existence results for the initial-boundary value problems in  $\mathbb{R}^n$ , are powerful enough to work in general Riemannian manifolds.

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## 1. INTRODUCTION

In this paper we are concerned with deriving *a priori* second order estimates for fully nonlinear parabolic equations on Riemannian manifolds. Let  $M^n$  be a compact Riemannian manifold of dimension  $n \geq 2$  with smooth boundary  $\partial M$  which may be empty ( $M$  is closed). Let  $\chi$  be a smooth  $(0, 2)$  tensor on  $\bar{M} = M \cup \partial M$  and  $f$  a smooth symmetric function of  $n$  variables. We consider the fully nonlinear parabolic equation

$$(1.1) \quad f(\lambda(\nabla^2 u + \chi)) = e^{u_t + \psi} \text{ in } M \times \{t > 0\},$$

where  $\nabla^2 u$  denotes the spatial Hessian of  $u$ ,  $u_t = \partial u / \partial t$ , and  $\lambda(A) = (\lambda_1, \dots, \lambda_n)$  will be the eigenvalues of a  $(0, 2)$  tensor  $A$ ; throughout the paper we shall use  $\nabla$  to denote the Levi-Civita connection of  $(M^n, g)$ , and assume  $\psi \in C^\infty(\bar{M} \times \{t \geq 0\})$ .

The corresponding elliptic equations were first studied by Caffarelli, Nirenberg and Spruck [1] in  $\mathbb{R}^n$ , as well as in [2], [4], [5], [6], [7], [8], [11], [14], [16] and [17] etc. Following [1], we assume  $f$  to be defined in an open symmetric convex cone  $\Gamma \subset \mathbb{R}^n$  with vertex at origin,  $\Gamma_n := \{\lambda \in \mathbb{R}^n : \lambda_i > 0, \forall 1 \leq i \leq n\} \subseteq \Gamma$ , and to satisfy the fundamental structure conditions which have become standard in the literature:

$$(1.2) \quad f_i = f_{\lambda_i} \equiv \frac{\partial f}{\partial \lambda_i} > 0 \text{ in } \Gamma, \quad 1 \leq i \leq n,$$

and

$$(1.3) \quad f \text{ is a concave function in } \Gamma.$$

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Equation (1.1) is parabolic for a solution  $u$  with  $\lambda[u] := \lambda(\nabla^2 u + \chi) \in \Gamma$  for  $x \in M$  and  $t > 0$  (see [1]); we shall call such functions *admissible*. It is uniformly parabolic if  $\lambda[u]$  falls in a compact subset of  $\Gamma$  and, on the other hand, may become degenerate if  $\lambda[u] \in \bar{\Gamma} = \Gamma \cup \partial\Gamma$ . To prevent the degeneracy we shall need the following condition

$$(1.4) \quad \sup_{\partial\Gamma} f := \sup_{\lambda_0 \in \partial\Gamma} \lim_{\lambda \rightarrow \lambda_0} f(\lambda) \leq 0.$$

In addition, we shall assume that  $f$  is unbounded from above. In particular,

$$(1.5) \quad \lim_{R \rightarrow \infty} f(R\mathbf{1}) = \infty.$$

where and hereafter  $\mathbf{1} = (1, \dots, 1)$ .

Throughout the paper, let  $\varphi^b \in C^\infty(\bar{M})$  with

$$(1.6) \quad \lambda[\varphi^b] \in \Gamma, \quad f(\lambda[\varphi^b]) > 0 \text{ in } \bar{M}$$

and, when  $\partial M \neq \emptyset$ ,  $\varphi^s \in C^\infty(\partial M \times \{t \geq 0\})$ . By the short time existence theorem, there exists a unique admissible solution  $u \in C^\infty(\bar{M} \times (0, t_0]) \cap C^0(M \times [0, t_0])$ , for some  $t_0 > 0$ , of equation (1.1) satisfying the initial boundary value conditions

$$(1.7) \quad u|_{t=0} = \varphi^b \text{ in } \bar{M}, \quad u = \varphi^s \text{ on } \partial M \times \{t > 0\}.$$

Moreover,  $u \in C^\infty(\bar{M} \times [0, t_0])$  if the following compatibility conditions are satisfied

$$(1.8) \quad f(\lambda[\varphi^b]) = e^{\varphi_t^s + \psi}, \quad \varphi^s = \varphi^b \text{ on } \partial M \times \{t = 0\}.$$

Our primary goal in this paper is to establish second order estimates for admissible solutions of the initial-boundary value problem (1.1) and (1.7). Without loss of generality, we may assume (1.8) is satisfied. For we only have to consider a new initial time, say  $t = t_0/2$  in place of  $t = 0$ , if necessary.

For  $T > 0$  let

$$M_T = M \times (0, T], \quad \bar{M}_T = \bar{M} \times (0, T]$$

and let  $\partial M_T := \partial_s M_T \cup \partial_b M_T$  be the parabolic boundary of  $M_T$  where

$$\partial_s M_T = \partial M \times [0, T], \quad \partial_b M_T = \bar{M} \times \{t = 0\}.$$

So  $\partial M_T = \partial_b M_T$  when  $M$  is closed. Let  $u \in C^{4,2}(M_T) \cap C^{2,1}(\bar{M}_T)$  be an admissible solution of the problem (1.1) and (1.7). We wish to establish the *a priori* estimate

$$(1.9) \quad |\nabla^2 u| \leq C \text{ in } \bar{M}_T.$$

As our first main result in this paper we derive (1.9) assuming the existence of an admissible subsolution.

**Theorem 1.1.** *In addition to conditions (1.2)-(1.5), suppose that there exists an admissible subsolution  $\underline{u} \in C^{2,1}(\bar{M}_T)$  satisfying*

$$(1.10) \quad f(\lambda[\underline{u}]) \geq e^{\underline{u}_t + \psi} \text{ in } M_T$$

and the initial-boundary conditions

$$(1.11) \quad \begin{cases} \underline{u} \leq \varphi^b & \text{on } \partial_b M_T, \\ \underline{u} = \varphi^s & \text{on } \partial_s M_T. \end{cases}$$

Then

$$(1.12) \quad \sup_{M_T} |\nabla^2 u| \leq C_1 + C_1 \max_{\partial M_T} |\nabla^2 u|$$

In particular, (1.9) holds when  $M$  is closed.

Suppose moreover that for any  $b > a > 0$ , there exists  $K_1 \geq 0$  such that

$$(1.13) \quad \sum f_i(\lambda) \lambda_i \geq -K_1 \left(1 + \sum f_i\right) \quad \text{in } \Gamma^{[a,b]} := \{\lambda \in \Gamma : a \leq f(\lambda) \leq b\}.$$

Then

$$(1.14) \quad \max_{\partial M_T} |\nabla^2 u| \leq C_2.$$

*Remark 1.2.* In Theorem 1.1 and the rest of this paper, unless otherwise indicated the constant  $C_1$  in (1.12) will depend on

$$(1.15) \quad |u|_{C^1(\overline{M_T})}, |\psi|_{C^{2,1}(\overline{M_T})}, |\underline{u}|_{C^{2,1}(\overline{M_T})}, \inf_{M_T} \text{dist}(\lambda[\underline{u}], \partial\Gamma),$$

as well as geometric quantities of  $M$ , while  $C_2$  in (1.14) will depend in addition on  $|\varphi^b|_{C^2(\overline{M})}$ ,  $|\varphi^s|_{C^{4,1}(\partial_s M_T)}$  and geometric quantities of  $\partial M$ .

*Remark 1.3.* The proof of (1.12) does not need assumptions (1.4) and (1.11). This will be clear in Section 2. For the boundary estimate (1.14), we need condition (1.4) to prevent equation (1.1) from being degenerate along the boundary. It would be interesting to establish (1.14) in the degenerate case. We also expect Theorem 1.1 to hold without conditions (1.5) and (1.13) which are fairly mild and technical in nature. When  $M$  is a bounded smooth domain in  $\mathbb{R}^n$  these assumptions can be removed.

*Remark 1.4.* If we replace (1.5) by the assumption

$$(1.16) \quad \lim_{|\lambda| \rightarrow \infty} |\lambda|^2 \sum f_i = \infty,$$

then  $C_1$  in (1.12) can be chosen independent of  $|u_t|_{C^0(\overline{M_T})}$ ; see Remark 2.4.

Our next result concerns (1.12) under a new condition which is optimal in many cases and is in general weaker than the subsolution assumption in Theorem 1.1, especially on closed manifolds. It is motivated by recent work in [6].

For  $\sigma \in \mathbb{R}$  define

$$\Sigma^\sigma := \{(\lambda, z) \in \Gamma \times \mathbb{R} : f(\lambda) > e^{z+\sigma}\}$$

and let  $\partial\Sigma^\sigma$  be the boundary of  $\Sigma^\sigma$ . By (1.2) and (1.3),  $\partial\Sigma^\sigma$  is a smooth convex hypersurface in  $\Gamma \times \mathbb{R}$ . For  $\hat{\lambda} = (\lambda, z) \in \partial\Sigma^\sigma$  let

$$\nu_{\hat{\lambda}} = \frac{(Df(\lambda), -f(\lambda))}{\sqrt{f(\lambda)^2 + |Df(\lambda)|^2}}$$

denote the unit normal vector to  $\partial\Sigma^\sigma$  at  $\hat{\lambda}$ . Finally, for  $\hat{\mu} \in \Gamma \times \mathbb{R}$  let

$$\hat{S}_{\hat{\mu}}^\sigma := \{\hat{\lambda} \in \partial\Sigma^\sigma : (\hat{\mu} - \hat{\lambda}) \cdot \nu_{\hat{\lambda}} \leq 0\}.$$

**Theorem 1.5.** *Under conditions (1.2) and (1.3), the estimate (1.12) holds provided that there exists an admissible function  $\underline{u} \in C^{2,1}(\overline{M_T})$  satisfying*

$$(1.17) \quad \hat{S}_{\hat{\mu}}^{\psi(x,t)} \cap \Gamma \times [a, b] \text{ is compact, } \forall (x, t) \in M_T, \forall [a, b] \subset \mathbb{R}$$

where  $\hat{\mu} = (\lambda[\underline{u}(x, t)], \underline{u}_t(x, t))$ .

By the concavity of  $f$ , if  $\underline{u}$  is an admissible subsolution then  $(\hat{\mu} - \hat{\lambda}) \cdot \nu_{\hat{\lambda}} \geq 0$  for any  $\hat{\lambda} \in \Sigma^{\psi(x,t)}$ .

*Remark 1.6.* In Theorems 1.1 and 1.5, the constants  $C_1$  and  $C_2$  depend on  $T$  only implicitly. For instance, if the quantities listed in (1.15) are all independent of  $T$ , then so is  $C_1$ . The independence on  $T$  of the estimates is important to understanding the asymptotic behaviors of solutions as  $t$  goes to infinity. If one allows  $C_1$  to depend on  $T$  (explicitly), (1.12) can be derived under much weaker conditions, and more easily.

**Theorem 1.7.** *Under assumptions (1.2), (1.3) and (1.6),*

$$(1.18) \quad |\nabla^2 u(x, t)| \leq Ce^{Bt} \left(1 + \max_{\partial M_T} |\nabla^2 u|\right), \quad \forall (x, t) \in M_T$$

where  $C$  and  $B$  depend on  $|\nabla u|_{C^0(\overline{M_T})}$ ,  $|\varphi^b|_{C^2(\bar{M})}$  and other known data. In particular, if  $M$  is closed then  $|\nabla^2 u(x, t)| \leq Ce^{Bt}$ .

Note that by (1.6) the function

$$\underline{u} := \varphi^b + t \min_{\bar{M}} \{\log f(\lambda[\varphi^b]) - \psi\}$$

is admissible and satisfies (1.10).

An immediate consequence of Theorem 1.7 is the following characterization of finite time blow-up solutions on closed manifolds.

**Corollary 1.8.** *Assume  $M$  is closed and  $f$  satisfies (1.2)-(1.4). Then equation (1.1) admits a unique admissible solution  $u \in C^\infty(M \times \mathbb{R}^+)$  with initial value function  $\varphi^b$  satisfying (1.6), provided that the a priori gradient estimate holds*

$$(1.19) \quad \sup_{M_T} |\nabla u| \leq C, \quad \forall T > 0$$

where  $C$  may depend on  $T$ . In other words, if  $u$  has a finite time blow-up at  $T < \infty$ , then

$$\lim_{t \rightarrow T^-} \max_{x \in M} |\nabla u(x, t)| = \infty.$$

So the long time existence of solutions in  $0 \leq t < \infty$  reduces to establishing gradient estimate (1.19). This is also true when  $\partial M \neq \emptyset$ . Using Theorem 1.1 we can prove the following existence results.

**Theorem 1.9.** *Assume (1.2)-(1.6), (1.13), and (1.10)-(1.11) hold for  $T \in (0, \infty]$ . Then there exists a unique admissible solution  $u \in C^\infty(\bar{M}_T) \cap C^0(\bar{M}_T)$  of equation (1.1) satisfying (1.7), provided that any one of the following conditions holds: (i)  $\Gamma = \Gamma_n$ ; (ii)  $(M, g)$  has nonnegative sectional curvature; (iii) there is  $\delta_0 > 0$  such that*

$$(1.20) \quad f_j \geq \delta_0 \sum f_i \text{ if } \lambda_j < 0, \text{ on } \partial \Gamma^\sigma \quad \forall \sigma > 0;$$

and (iv)  $K_1 = 0$  in (1.13) and  $\nabla^2 w \geq \chi$  for some function  $w \in C^2(\bar{M})$ .

The assumptions (i)-(iv) are only needed in deriving the gradient estimates. It would be interesting to remove these assumptions. When  $\partial M = \emptyset$ , Theorem 1.9 holds without assumptions (1.5) and (1.10)-(1.11), and condition (1.13) can be removed in each of the cases (i)-(iii).

Theorem 1.9 applies to a very general class of equations including  $f = \sigma_k^{1/k}$  and  $f = (\sigma_k/\sigma_l)^{1/(k-l)}$ ,  $1 \leq l < k \leq n$  where  $\sigma_k$  is the  $k$ -th elementary symmetric function defined on the cone  $\Gamma_k := \{\lambda \in \mathbb{R}^n : \sigma_j(\lambda) > 0, \forall 1 \leq j \leq k\}$ . Another interesting example is  $f = \log P_k$  to which Theorem 1.9 applies, where

$$P_k(\lambda) := \prod_{i_1 < \dots < i_k} (\lambda_{i_1} + \dots + \lambda_{i_k}), \quad 1 \leq k \leq n$$

defined in the cone

$$\mathcal{P}_k := \{\lambda \in \mathbb{R}^n : \lambda_{i_1} + \dots + \lambda_{i_k} > 0, \forall 1 \leq i_1 < \dots < i_k \leq n\}.$$

**Corollary 1.10.** *Let  $f = (\sigma_k/\sigma_l)^{1/(k-l)}$ ,  $\Gamma = \Gamma_k$ , ( $0 \leq l < k \leq n$ ;  $\sigma_0 = 1$ ), or  $f = \log P_k$  and  $\Gamma = \mathcal{P}_k$ . The parabolic problem (1.1) and (1.7) with smooth data admits a unique admissible solution  $u \in C^\infty(\bar{M}_T) \cap C^0(\bar{M}_T)$ , provided that there exists an admissible subsolution  $\underline{u} \in C^{2,1}(\bar{M}_T)$  satisfying (1.10)-(1.11).*

For  $f = (\sigma_k/\sigma_l)^{1/(k-l)}$  or  $f = \log P_k$ , an admissible subsolution satisfies (1.17); see [6]. Except for  $f = \sigma_k^{1/k}$ , Corollary 1.10 is new even when  $M$  is a bounded smooth domain in  $\mathbb{R}^n$ ; see also [12]. On the other hand, for a bounded smooth domain in  $\mathbb{R}^n$ , we have the following result which is essentially optimal, both in terms of assumptions on  $f$  and the generality of the domain.

**Theorem 1.11.** *Let  $M$  be a bounded smooth domain in  $\mathbb{R}^n$ ,  $0 < T \leq \infty$ , and let  $\chi = \{\chi_{ij}\}$  be a symmetric matrix with  $\chi_{ij} \in C^\infty(\bar{M}_T)$ . Under conditions (1.2)-(1.6)*

and (1.10)-(1.11), there exists a unique admissible solution  $u \in C^\infty(\bar{M}_T) \cap C^0(\overline{M_T})$  of equation (1.1) satisfying (1.7).

The first initial-boundary value problem for equation (1.1) or (1.21) in  $\mathbb{R}^n$  was treated by Ivochinkina-Ladyzhenskaya [9], [10], and by Wang [18], Chou-Wang [2] for  $f = (\sigma_k)^{1/k}$ ; see also [15]. Jiao-Sui [12] recently studied equation (1.21) on Riemannian manifolds under additional assumptions.

The rest of the article is divided into three sections. In Sections 2 and 3 we derive (1.12) and (1.14) respectively, completing the proofs of Theorems 1.1, 1.5 and 1.7. Instead of (1.1), we shall deal with the equation

$$(1.21) \quad f(\lambda(\nabla^2 u + \chi)) = u_t + \psi$$

under essentially the same assumptions on  $f$  with the exception that (1.4) is replaced by

$$(1.22) \quad \inf_{\partial_s M_T} (\varphi_t + \psi) - \sup_{\partial \Gamma} f > 0$$

which is need in the proof of (1.14). Accordingly, the functions  $\varphi^b$  and  $\underline{u} \in C^{2,1}(\overline{M_T})$  are assumed to satisfy  $\lambda[\varphi^b] \in \Gamma$  in  $\bar{M}$  and, respectively,

$$(1.23) \quad f(\lambda[\underline{u}]) \geq \underline{u}_t + \psi \text{ in } M_T$$

in place of (1.10). Note that if  $f > 0$  in  $\Gamma$  and satisfies (1.2), (1.3), (1.5) and (1.13) then the function  $\log f$  still satisfies theses assumptions. So equation (1.1) is covered by (1.21) in most cases, and we shall derive the estimates for equation (1.21).

In Section 4 we briefly discuss the proof of the existence results and the preliminary estimates needed in the proof.

At the end of this Introduction we recall the following commonly used notations

$$\begin{aligned} |u|_{C^{k,l}(\overline{M_T})} &= \sum_{j=0}^k |\nabla^j u|_{C^0(\overline{M_T})} + \sum_{j=1}^l \left| \frac{\partial^j u}{\partial t^j} \right|_{C^0(\overline{M_T})}, \\ |u|_{C^{k+\alpha, l+\beta}(\overline{M_T})} &= |u|_{C^{k,l}(\overline{M_T})} + |\nabla^k u|_{C^\alpha(\overline{M_T})} + \left| \frac{\partial^l u}{\partial t^l} \right|_{C^\beta(\overline{M_T})} \end{aligned}$$

where  $0 < \alpha, \beta < 1$  and  $k, l = 1, 2, \dots$ , for a function  $u$  sufficiently smooth on  $\overline{M_T}$ . We shall also write  $|u|_{C^k(\overline{M_T})} = |u|_{C^{k,k}(\overline{M_T})}$ .

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## 2. GLOBAL ESTIMATES FOR SECOND DERIVATIVES

A substantial difficulty in deriving the global estimate (1.12), which is our primary goal in this section, is caused due to the presense of curvature of  $M$ ; another is the lack of (globally defined) functions or geometric quantities with desirable properties. In our proof the use of the function  $\underline{u}$ , which is either an admissible subsolution as in Theorem 1.1 or satisfies (1.17), is critical. We shall consider equation (1.21) in place of (1.1).

Let  $u \in C^{4,2}(M_T) \cap C^{2,1}(\overline{M_T})$  be an admissible solution of (1.21), and  $\underline{u} \in C^{2,1}(\overline{M_T})$  an admissible function. We assume that  $u$  admits an *a priori*  $C^1$  bound

$$(2.1) \quad |u|_{C^1(\overline{M_T})} \leq C.$$

Let  $\phi(s) = -\log(1 - bs^2)$  and

$$(2.2) \quad \eta = \phi(1 + |\nabla(u - \underline{u})|^2) + a(\underline{u} - u - \delta t)$$

where  $a, b, \delta > 0$  are constants and  $\underline{u} \in C^{2,1}(\overline{M_T})$  is an admissible function; we shall choose  $\delta = 1$  or  $0$ ,  $a$  sufficiently large while  $b$  small enough,

$$(2.3) \quad b \leq \frac{1}{8b_1^2}, \quad b_1 = 1 + \sup_{M_T} |\nabla(u - \underline{u})|^2.$$

Consider the quantity

$$W = \sup_{(x,t) \in M_T} \max_{\xi \in T_x M^n, |\xi|=1} (\nabla_{\xi\xi} u + \chi(\xi, \xi)) e^\eta.$$

Suppose  $W$  is achieved at an interior point  $(x_0, t_0) \in M_T$  for a unit vector  $\xi \in T_{x_0} M^n$ . Let  $e_1, \dots, e_n$  be smooth orthonormal local frames about  $x_0$  such that  $e_1 = \xi$ ,  $\nabla_i e_j = 0$  and  $U_{ij} := \nabla_{ij} u + \chi_{ij}$  are diagonal at  $(x_0, t_0)$ . So  $W = U_{11}(x_0, t_0) e^{\eta(x_0, t_0)}$ . We wish to derive a bound

$$(2.4) \quad U_{11}(x_0, t_0) \leq C.$$

Write equation (1.21) in the form

$$(2.5) \quad u_t = F(U) - \psi, \quad U = \{U_{ij}\}$$

where  $F$  is defined by

$$F(A) \equiv f(\lambda[A])$$

for an  $n \times n$  symmetric matrices  $A = \{A_{ij}\}$  with eigenvalues  $\lambda[A] \in \Gamma$ . Differentiating (2.5) gives

$$(2.6) \quad \begin{aligned} u_{tt} &= F^{ij} U_{ijt} - \psi_t, \\ \nabla_k u_t &= F^{ij} \nabla_k U_{ij} - \nabla_k \psi, \quad \forall k, \\ \nabla_{11} u_t &= F^{ij} \nabla_{11} U_{ij} + F^{ij,kl} \nabla_1 U_{ij} \nabla_1 U_{kl} - \nabla_{11} \psi. \end{aligned}$$

Throughout the paper we use the notation

$$F^{ij} = \frac{\partial F}{\partial A_{ij}}(U), \quad F^{ij,kl} = \frac{\partial^2 F}{\partial A_{ij} \partial A_{kl}}(U).$$

The matrix  $\{F^{ij}\}$  has eigenvalues  $f_1, \dots, f_n$ , and therefore is positive definite when  $f$  satisfies (1.2), while (1.3) implies that  $F$  is a concave function; see [1]. Moreover, the following identities hold

$$F^{ij}U_{ij} = \sum f_i \lambda_i, \quad F^{ij}U_{ik}U_{kj} = \sum f_i \lambda_i^2.$$

We also note that  $F^{ij}$  are diagonal at  $(x_0, t_0)$ .

**Proposition 2.1.** *For any  $a, C_1 > 0$  there exists a constant  $b > 0$  satisfying (2.3) such that, at  $(x_0, t_0)$ , if  $U_{11} \geq C_1 a/b$  then*

$$(2.7) \quad \frac{b}{2} F^{ii} U_{ii}^2 + a F^{ii} \nabla_{ii}(\underline{u} - u) - a(\underline{u}_t - u_t) + a\delta \leq C \sum F^{ii} + C.$$

*Proof.* We shall assume  $U_{11}(x_0, t_0) \geq 1$ . At  $(x_0, t_0)$  where the function  $\log U_{11} + \eta$  has its maximum,

$$(2.8) \quad \frac{(\nabla_{11} u)_t}{U_{11}} + \eta_t \geq 0, \quad \frac{\nabla_i U_{11}}{U_{11}} + \nabla_i \eta = 0, \quad 1 \leq i \leq n,$$

and

$$(2.9) \quad \frac{1}{U_{11}} F^{ii} \nabla_{ii} U_{11} - \frac{1}{U_{11}^2} F^{ii} (\nabla_i U_{11})^2 + F^{ii} \nabla_{ii} \eta \leq 0.$$

From the identity

$$(2.10) \quad \begin{aligned} \nabla_{ijkl} v - \nabla_{klij} v &= R_{ljk}^m \nabla_{im} v + \nabla_i R_{ljk}^m \nabla_m v + R_{lik}^m \nabla_{jm} v \\ &\quad + R_{jik}^m \nabla_{lm} v + R_{jil}^m \nabla_{km} v + \nabla_k R_{jil}^m \nabla_m v \end{aligned}$$

it follows that

$$(2.11) \quad F^{ii} \nabla_{ii} U_{11} \geq F^{ii} \nabla_{11} U_{ii} - C U_{11} \sum F^{ii}$$

where  $C$  depends on  $|\nabla u|_{C^0(\bar{M}_T)}$  and geometric quantities of  $M$ . By (2.9), (2.11) and (2.6) we obtain

$$(2.12) \quad \begin{aligned} F^{ii} \nabla_{ii} \eta - \eta_t &\leq \frac{1}{U_{11}} F^{ij,kl} \nabla_1 U_{ij} \nabla_1 U_{kl} + \frac{1}{U_{11}^2} F^{ii} (\nabla_i U_{11})^2 \\ &\quad - \frac{\nabla_{11} \psi}{U_{11}} + C \sum F^{ii}. \end{aligned}$$

Let

$$J = \{i : 3U_{ii} \leq -U_{11}\}, \quad K = \{i > 1 : 3U_{ii} > -U_{11}\}.$$

As in [6], which uses an idea of Urbas [17], one derives

$$(2.13) \quad F^{ii} \nabla_{ii} \eta - \eta_t \leq \sum_{i \in J} F^{ii} (\nabla_i \eta)^2 + C F^{11} \sum_{i \notin J} (\nabla_i \eta)^2 - \frac{\nabla_{11} \psi}{U_{11}} + C \sum F^{ii}.$$



For convenience we write  $w = \underline{u} - u$ ,  $s = 1 + |\nabla w|^2$ , and calculate

$$\begin{aligned}\nabla_i \eta &= 2\phi' \nabla_k w \nabla_{ik} w + a \nabla_i w, \\ \eta_t &= 2\phi' \nabla_k w (\nabla_k w)_t + a w_t - a\delta, \\ \nabla_{ii} \eta &= 2\phi' (\nabla_{ik} w \nabla_{ik} w + \nabla_k w \nabla_{iik} w) + 4\phi'' (\nabla_k w \nabla_{ik} w)^2 + a \nabla_{ii} w,\end{aligned}$$

while

$$\phi'(s) = \frac{2bs}{1-bs^2}, \quad \phi''(s) = \frac{2b+2b^2s^2}{(1-bs^2)^2} > 4(\phi')^2.$$

Hence,

$$(2.14) \quad \sum_{i \in J} F^{ii} (\nabla_i \eta)^2 \leq 8(\phi')^2 \sum_{i \in J} F^{ii} (\nabla_k w \nabla_{ik} w)^2 + 2|\nabla w|^2 a^2 \sum_{i \in J} F^{ii},$$

and

$$(2.15) \quad \sum_{i \notin J} (\nabla_i \eta)^2 \leq Ca^2 + C(\phi')^2 U_{11}^2.$$

By (2.6),

$$(2.16) \quad \begin{aligned}F^{ii} \nabla_{ii} \eta - \eta_t &\geq \phi' F^{ii} U_{ii}^2 + 2\phi'' F^{ii} (\nabla_k w \nabla_{ik} w)^2 \\ &\quad + a F^{ii} \nabla_{ii} w - a w_t + a\delta - C\phi' \left(1 + \sum F^{ii}\right).\end{aligned}$$

It follows from (2.13)-(2.16) that

$$(2.17) \quad \begin{aligned}&\phi' F^{ii} U_{ii}^2 + a F^{ii} \nabla_{ii} w - a w_t + a\delta \\ &\leq Ca^2 \sum_{i \in J} F^{ii} + C(a^2 + (\phi')^2 U_{11}^2) F^{11} - \frac{\nabla_{11} \psi}{U_{11}} + C\left(\phi' + \sum F^{ii}\right).\end{aligned}$$

Note that

$$(2.18) \quad F^{ii} U_{ii}^2 \geq F^{11} U_{11}^2 + \sum_{i \in J} F^{ii} U_{ii}^2 \geq F^{11} U_{11}^2 + \frac{U_{11}^2}{9} \sum_{i \in J} F^{ii}.$$

We may fix  $b$  small to derive (2.7) when  $U_{11} \geq Ca/b$ .  $\square$

To proceed we need the following lemma which is key to the proof of Theorem 1.1, both for (1.12) in this section and (1.14) in the next section; compare with Lemma 2.1 in [7].

**Lemma 2.2.** *Let  $K$  be a compact subset of  $\Gamma$  and  $\beta > 0$ . There is constant  $\varepsilon > 0$  such that, for any  $\mu \in K$  and  $\lambda \in \Gamma$ , when  $|\nu_\mu - \nu_\lambda| \geq \beta$  (where  $\nu_\lambda = Df(\lambda)/|Df(\lambda)|$  denotes the unit normal vector to the level surface of  $f$  through  $\lambda$ ),*

$$(2.19) \quad \sum f_i(\lambda)(\mu_i - \lambda_i) \geq f(\mu) - f(\lambda) + \varepsilon \left(1 + \sum f_i(\lambda)\right).$$

*Proof.* Since  $\nu_\mu$  is smooth in  $\mu \in \Gamma$  and  $K$  is compact, there is  $\epsilon_0 > 0$  such that for any  $0 \leq \epsilon \leq \epsilon_0$ ,

$$K^\epsilon := \{\mu^\epsilon := \mu - \epsilon \mathbf{1} : \mu \in K\}$$

is still a compact subset of  $\Gamma$  and

$$|\nu_\mu - \nu_{\mu^\epsilon}| \leq \frac{\beta}{2}, \quad \forall \mu \in K.$$

Consequently, if  $\mu \in K$  and  $\lambda \in \Gamma$  satisfy  $|\nu_\mu - \nu_\lambda| \geq \beta$  then  $|\nu_{\mu^\epsilon} - \nu_\lambda| \geq \frac{\beta}{2}$ .

By the smoothness of the level surfaces of  $f$ , there exists  $\delta > 0$  (which depends on  $\beta$  but is uniform in  $\epsilon \in [0, \epsilon_0]$ ) such that

$$\min_{\mu \in K} \min_{0 \leq \epsilon \leq \epsilon_0} \text{dist}(\partial B_{\delta}^{\frac{\beta}{2}}(\mu^\epsilon), \partial \Gamma^{f(\mu^\epsilon)}) > 0$$

where  $\partial B_{\delta}^{\frac{\beta}{2}}(\mu^\epsilon)$  denotes the spherical cap

$$\partial B_{\delta}^{\frac{\beta}{2}}(\mu^\epsilon) = \left\{ \zeta \in \partial B_{\delta}(\mu^\epsilon) : \nu_{\mu^\epsilon} \cdot (\zeta - \mu^\epsilon) / \delta \geq \frac{\beta}{2} \sqrt{1 - \beta^2/16} \right\}.$$

Therefore,

$$(2.20) \quad \theta \equiv \min_{\mu \in K} \min_{0 \leq \epsilon \leq \epsilon_0} \min_{\zeta \in \partial B_{\delta}^{\frac{\beta}{2}}(\mu^\epsilon)} \{f(\zeta) - f(\mu^\epsilon)\} > 0.$$

Let  $P$  be the two-plane through  $\mu^\epsilon$  spanned by  $\nu_{\mu^\epsilon}$  and  $\nu_\lambda$  (translated to  $\mu^\epsilon$ ), and  $L$  the line on  $P$  through  $\mu^\epsilon$  and perpendicular to  $\nu_\lambda$ . Since  $0 < \nu_{\mu^\epsilon} \cdot \nu_\lambda \leq 1 - \beta^2/8$ ,  $L$  intersects  $\partial B_{\delta}^{\frac{\beta}{2}}(\mu^\epsilon)$  at a unique point  $\zeta$ . By the concavity of  $f$  we see that ,

$$(2.21) \quad \begin{aligned} \sum f_i(\lambda)(\mu_i^\epsilon - \lambda_i) &= \sum f_i(\lambda)(\zeta_i - \lambda_i) \\ &\geq f(\zeta) - f(\lambda) \\ &\geq \theta + f(\mu^\epsilon) - f(\lambda), \quad \forall 0 \leq \epsilon \leq \epsilon_0. \end{aligned}$$

Next, by the continuity of  $f$  we may choose  $0 < \epsilon_1 \leq \epsilon_0$  with  $|f(\mu^{\epsilon_1}) - f(\mu)| \leq \frac{1}{2}\theta$ . Hence

$$(2.22) \quad \sum f_i(\lambda)(\mu_i - \epsilon_1 - \lambda_i) \geq f(\mu) - f(\lambda) + \frac{1}{2}\theta.$$

This proves (2.19) with  $\varepsilon = \min\{\theta/2, \epsilon_1\}$ . □

*Remark 2.3.* Alternatively, one can first prove

$$\sum f_i(\lambda)(\mu_i - \lambda_i) \geq \theta + f(\mu) - f(\lambda).$$

Then choose  $\epsilon > 0$  small such that  $0 \leq f(\mu) - f(\mu^\epsilon) \leq \frac{\theta}{2}$ . By the concavity of  $f$ ,

$$(2.23) \quad \sum f_i(\lambda)(\mu_i^\epsilon - \lambda_i) \geq f(\mu^\epsilon) - f(\lambda) \geq f(\mu) - f(\lambda) - \frac{\theta}{2}.$$

Now add these two inequalities to obtain (2.19).

We now continue to prove (2.4). Assume first that  $\underline{u}$  is a subsolution, i.e.  $\underline{u}$  satisfies (1.23). Since  $\lambda[\underline{u}]$  falls in a compact subset of  $\Gamma$ ,

$$(2.24) \quad \beta := \frac{1}{2} \min \text{dist}(\nu_{\lambda[\underline{u}]}, \partial\Gamma_n) > 0.$$

Let  $\lambda = \lambda[u](x_0, t_0)$  and  $\mu = \lambda[\underline{u}](x_0, t_0)$ . If  $|\nu_\mu - \nu_\lambda| \geq \beta$  then by Lemma 2.2,

$$(2.25) \quad F^{ii} \nabla_{ii} w - w_t \geq \sum f_i(\lambda)(\mu_i - \lambda_i) - f(\mu) - f(\lambda) \geq \varepsilon \left(1 + \sum F^{ii}\right).$$

The first inequality follows from Lemma 6.2 in [1]; see [6]. We may fix  $a$  sufficiently large to derive a bound  $U_{11}(x_0, t_0) \leq C$  by (2.7).

Suppose now that  $|\nu_\mu - \nu_\lambda| < \beta$  and therefore  $\nu_\lambda - \beta \mathbf{1} \in \Gamma_n$ . It follows that

$$(2.26) \quad F^{ii} \geq \frac{\beta}{\sqrt{n}} \sum F^{kk}, \quad \forall 1 \leq i \leq n.$$

Since  $\underline{u}$  is a subsolution,  $F^{ii} \nabla_{ii} w - w_t \geq 0$  by the concavity of  $f$ . By (2.7) and (2.26) we obtain

$$(2.27) \quad \frac{b\beta}{2\sqrt{n}} U_{11}^2 \sum F^{ii} + a\delta \leq C \sum F^{ii} + C.$$

If we allow  $\delta = 1$ , a bound  $U_{11}(x_0, t_0) \leq C$  would follow when  $a$  is sufficiently large without using assumption (1.5). This gives (1.18) in Theorem 1.7.

For the case  $\delta = 0$ , we need assumption (1.5). First, by the concavity of  $f$ ,

$$(2.28) \quad \begin{aligned} |\lambda| \sum f_i &\geq f(|\lambda| \mathbf{1}) - f(\lambda) + \sum f_i \lambda_i \\ &\geq f(|\lambda| \mathbf{1}) - f(\lambda) - \frac{1}{4|\lambda|} \sum f_i \lambda_i^2 - |\lambda| \sum f_i. \end{aligned}$$

Hence, by assumption (1.5),

$$(2.29) \quad \begin{aligned} U_{11}^2 \sum F^{ii} &\geq \frac{U_{11}}{2n} (f(U_{11} \mathbf{1}) - u_t - \psi) - \frac{1}{8} \sum F^{ii} U_{ii}^2 \\ &\geq \frac{U_{11}}{2n} - \frac{U_{11}^2}{8} \sum F^{ii} \end{aligned}$$

when  $U_{11}$  is sufficiently large. A bound  $U_{11}(x_0, t_0) \leq C$  therefore follows from (2.27). The proof of (1.12) in Theorem 1.1 is complete.

*Remark 2.4.* If (1.16) holds, a bound  $U_{11}(x_0, t_0) \leq C$  follows from (2.27) directly (without using (1.5)) and is independent to  $|u_t|_{C^0(\overline{M_T})}$ .

Suppose now that  $\underline{u}$  satisfies (1.17) with the obvious modification, i.e. with  $\Sigma^\sigma$  redefined as  $\Sigma^\sigma = \{(\lambda, p) \in \Gamma \times \mathbb{R} : f(\lambda) > p + \sigma\}$ . By Lemma 2.5 below we have

$$(2.30) \quad F^{ii} \nabla_{ii} w - w_t \geq \varepsilon \left(1 + \sum F^{ii}\right);$$

when  $U_{11}$  is sufficiently large. Therefore, fixing  $a$  large in (2.7) gives  $U_{11}(x_0, t_0) \leq C$ . This completes the proof of (1.12) in Theorem 1.5, subject to the proof of Lemma 2.5.

**Lemma 2.5.** *Let  $K$  be a compact subset of  $\Gamma \times \mathbb{R}$  such that  $\hat{S}_\mu^\sigma[a, b] := \hat{S}_\mu^\sigma \cap \Gamma \times [a, b]$  is compact for any  $\hat{\mu} \in K$ . Then there exist  $R, \epsilon > 0$  such that for all  $\lambda \in \Gamma^{[a, b]}$ , when  $|\lambda| > R$ ,*

$$(2.31) \quad \sum f_i(\lambda)(\mu_i - \lambda_i) \geq z - f(\lambda) + \varepsilon \left(1 + \sum f_i(\lambda)\right), \quad \forall (\mu, z) \in K.$$

In some sense this is a parabolic version of Theorem 2.17 in [6]. Its proof is long but follows similar ideas in [6]. So we include it in the Appendix for completeness and for the reader's convenience.

*Remark 2.6.* If  $\underline{u}$  is an admissible strict subsolution, i.e.

$$(2.32) \quad f(\lambda[\underline{u}]) \geq \underline{u}_t + \psi + \delta \quad \text{in } M_T$$

for some  $\delta > 0$ , then we can choose  $\epsilon > 0$  such that  $\lambda^\epsilon[\underline{u}] := \lambda[\underline{u}] - \epsilon \mathbf{1} \in \Gamma$  and

$$(2.33) \quad f(\lambda^\epsilon[\underline{u}]) \geq \underline{u}_t + \psi + \frac{\delta}{2} \quad \text{in } M_T.$$

By the concavity of  $f$  we see that

$$\sum f_i(\lambda[u])(\lambda_i^\epsilon[\underline{u}] - \lambda_i[u]) \geq f(\lambda^\epsilon[\underline{u}]) - f(\lambda[u]) \geq \underline{u}_t - u_t + \frac{\delta}{2}.$$

Therefore one can derive (2.4) directly from Proposition 2.1. This can be used to prove Theorem 1.7 as  $\underline{u} = \varphi^b + At$  is a strict subsolution of equation (1.21) for any constant  $A < \inf_M f(\lambda[\varphi^b]) - \sup_{M_T} \psi$ .

### 3. SECOND ORDER BOUNDARY ESTIMATES

Let  $u \in C^{3,1}(\overline{M_T})$  be an admissible solution of (1.21) satisfying (1.7) and the  $C^1$  estimate (2.1). In this section we derive (1.14) under the assumptions (1.2), (1.3), (1.5), (1.13) and (1.22) on  $f$ . Clearly we only need to focus on  $\partial_s M_T$ .

For a point  $x_0 \in \partial M$  we shall choose smooth orthonormal local frames  $e_1, \dots, e_n$  around  $x_0$  such that  $e_n$ , when restricted to  $\partial M$ , is the interior unit normal to  $\partial M$ . By the boundary condition  $u = \varphi^s$  on  $\partial_s M_T$  we obtain

$$(3.1) \quad |\nabla_{\alpha\beta} u(x_0, t_0)| \leq C, \quad \forall 1 \leq \alpha, \beta < n, \quad \forall 0 \leq t \leq T.$$

Let  $\rho(x)$  and  $d(x)$  denote the distances from  $x \in \bar{M}$  to  $x_0$  and  $\partial M$ , respectively. Let  $M_T^\delta = \{(x, t) \in M_T : \rho(x) < \delta\}$ , and  $\partial M_T^\delta$  be the parabolic boundary of  $M_T^\delta$ ,

$$\partial M_T^\delta = \overline{M_T^\delta} \setminus M_T^\delta.$$

We fix  $\delta_0 > 0$  sufficiently small such that both  $\rho$  and  $d$  are smooth in  $M_T^{\delta_0}$ . Let  $\mathcal{L}$  denote the linear parabolic operator

$$\mathcal{L}w = F^{ij} \nabla_{ij} w - w_t,$$

and

$$(3.2) \quad \Psi = A_1 v + A_2 \rho^2 - A_3 \sum_{l < n} |\nabla_l(u - \varphi)|^2$$

where

$$(3.3) \quad v = u - \underline{u} + sd - \frac{Nd^2}{2}$$

and  $\underline{u} \in C^{2,1}(\overline{M_T})$  is an admissible subsolution satisfying (1.23) and (1.11).

**Lemma 3.1.** *Assume (1.2), (1.3), (1.5), (1.13) hold and  $\underline{u}$  satisfies (1.23) and (1.11). Then for constant  $K > 0$ , there exist uniform positive constants  $s, \delta$  sufficiently small, and  $A_1, A_2, A_3, N$  sufficiently large such that  $\Psi \geq K(d + \rho^2)$  in  $M_T^\delta$  and*

$$(3.4) \quad \mathcal{L}\Psi \leq -K \left( 1 + \sum f_i |\lambda_i| + \sum f_i \right) \text{ in } M_T^\delta.$$

*Proof.* This is parabolic version of Lemma 3.1 in [7]. Since there are some substantial differences in several places, for completeness and reader's convenience we include a detailed proof.

First we note that  $\mathcal{L}(u - \underline{u}) \leq 0$  by the concavity of  $f$  and since  $\underline{u}$  is a subsolution, and by (2.6) ,

$$(3.5) \quad |\mathcal{L}\nabla_k(u - \varphi)| \leq C \left( 1 + \sum f_i |\lambda_i| + \sum f_i \right), \quad \forall 1 \leq k \leq n.$$

It follows that

$$(3.6) \quad \sum_{l < n} \mathcal{L}|\nabla_l(u - \varphi)|^2 \geq \sum_{l < n} F^{ij} U_{il} U_{jl} - C \left( 1 + \sum f_i |\lambda_i| + \sum f_i \right).$$

By Proposition 2.19 in [6] there exists an index  $r$  such that

$$(3.7) \quad \sum_{l < n} F^{ij} U_{il} U_{jl} \geq \frac{1}{2} \sum_{i \neq r} f_i \lambda_i^2.$$

At a fixed point  $(x, t)$ , denote  $\mu = \lambda(\nabla^2 \underline{u} + \chi)$  and  $\lambda = \lambda(\nabla^2 u + \chi)$ . As in Section 2 we consider two cases separately: **(a)**  $|\nu_\mu - \nu_\lambda| < \beta$  and **(b)**  $|\nu_\mu - \nu_\lambda| \geq \beta$ , where  $\beta$  is given in (2.24).

Case **(a)**  $|\nu_\mu - \nu_\lambda| < \beta$ . We have by (2.26)

$$(3.8) \quad f_i \geq \frac{\beta}{\sqrt{n}} \sum f_k, \quad \forall 1 \leq i \leq n.$$

We next show that this implies the following inequality for any index  $r$

$$(3.9) \quad \sum_{i \neq r} f_i \lambda_i^2 \geq c_0 \sum f_i \lambda_i^2 - C_0 \sum f_i$$

for some  $c_0, C_0 > 0$ .

Since  $\sum \lambda_i \geq 0$ , we see that

$$(3.10) \quad \sum_{\lambda_i < 0} \lambda_i^2 \leq \left( - \sum_{\lambda_i < 0} \lambda_i \right)^2 \leq n \sum_{\lambda_i > 0} \lambda_i^2.$$

Therefore, by (3.8) and (3.10) we obtain if  $\lambda_r < 0$ ,

$$f_r \lambda_r^2 \leq n f_r \sum_{\lambda_i > 0} \lambda_i^2 \leq \frac{n\sqrt{n}}{\beta} \sum_{\lambda_i > 0} f_i \lambda_i^2.$$

On the other hand, by the concavity of  $f$  and assumption (1.5) we have

$$(3.11) \quad \sum f_i(b - \lambda_i) \geq f(b\mathbf{1}) - f(\lambda) = f(b\mathbf{1}) - u_t - \psi \geq 1$$

for  $b > 0$  sufficiently large. It follows that if  $\lambda_r > 0$ ,

$$f_r \lambda_r \leq b \sum f_i - \sum_{\lambda_i < 0} f_i \lambda_i.$$

By (3.8) and Schwarz inequality,

$$\begin{aligned} \frac{\beta f_r \lambda_r^2}{\sqrt{n}} \sum f_k &\leq f_r^2 \lambda_r^2 \leq 2b^2 \left( \sum f_i \right)^2 + 2 \sum_{\lambda_k < 0} f_k \sum_{\lambda_i < 0} f_i \lambda_i^2 \\ &\leq 2 \left( \sum_{\lambda_i < 0} f_i \lambda_i^2 + b^2 \sum f_i \right) \sum f_k. \end{aligned}$$

This finishes the proof of (3.9).

Letting  $b = n|\lambda|$  in (3.11), we see that

$$(3.12) \quad (n+1)|\lambda| \sum f_i \geq \sum f_i(n|\lambda| - \lambda_i) \geq f(n|\lambda|\mathbf{1}) - f(\lambda) \geq 1,$$

and consequentley by (3.8),

$$(3.13) \quad \sum f_i \lambda_i^2 \geq \frac{\beta|\lambda|^2}{\sqrt{n}} \sum f_i \geq \frac{\beta|\lambda|}{(n+1)\sqrt{n}},$$

provided that  $|\lambda| \geq R$  for  $R$  sufficiently large.

It now follows from (3.6), (3.7), (3.9), (3.13) and Schwartz inequality that when  $|\lambda| \geq R$ ,

$$(3.14) \quad \sum_{l < n} \mathcal{L} |\nabla_l(u - \varphi)|^2 \geq c_1 \sum f_i \lambda_i^2 + 2c_1 |\lambda| - C - C_1 \sum f_i.$$

for some  $c_1, C_1 > 0$ . We now fix  $R \geq C/c_1$ .

Turning to the fuction  $v$ , we note that by (3.8),

$$\begin{aligned} \mathcal{L} v &\leq \mathcal{L}(u - \underline{u}) + C(s + Nd) \sum F^{ii} - NF^{ij} \nabla_i d \nabla_j d \\ (3.15) \quad &\leq \left( C(s + Nd) - \frac{\beta N}{\sqrt{n}} \right) \sum F^{ii} \end{aligned}$$

since  $\mathcal{L}(u - \underline{u}) \leq 0$  and  $|\nabla d| \equiv 1$ . For  $N$  sufficiently large we have

$$(3.16) \quad \mathcal{L}v \leq - \sum f_i \text{ in } M_T^\delta$$

and therefore, in view of (3.14) and (3.16),

$$(3.17) \quad \mathcal{L}\Psi \leq -A_3 c_1 \left( |\lambda| + \sum f_i \lambda_i^2 \right) + (-A_1 + CA_2 + C_1 A_3) \sum f_i.$$

when  $|\lambda| \geq R$ , for any  $s \in (0, 1]$  as long as  $\delta$  is sufficiently small. From now on  $A_3$  is fixed such that  $A_3 c_1 R \geq K$ , so  $A_3 \geq CK/c_1^2$ .

Suppose now that  $|\lambda| \leq R$ . By (1.2) and (1.3) we have

$$(3.18) \quad \begin{aligned} 2R \sum f_i &\geq \sum f_i \lambda_i + f(2R\mathbf{1}) - f(\lambda) \\ &\geq -R \sum f_i + f(2R\mathbf{1}) - f(R\mathbf{1}). \end{aligned}$$

Therefore,

$$\sum f_i \geq \frac{f(2R\mathbf{1}) - f(R\mathbf{1})}{3R} \equiv C_R > 0.$$

It follows from (2.26) that there is a uniform lower bound

$$(3.19) \quad f_i \geq \frac{\beta}{\sqrt{n}} \sum f_k \geq \frac{\beta C_R}{\sqrt{n}}, \quad \forall 1 \leq i \leq n.$$

Consequently, since  $|\nabla d| = 1$ ,

$$F^{ij} \nabla_i d \nabla_j d \geq \frac{\beta}{2\sqrt{n}} \left( C_R + \sum f_i \right).$$

From (3.15) we see that when  $\delta$  is sufficiently small and  $N$  sufficiently large,

$$(3.20) \quad \mathcal{L}v \leq - \left( 1 + \sum f_i \right) \text{ in } M_T^\delta.$$

Combining (3.6), (3.7), (3.9), (3.20) yields

$$(3.21) \quad \mathcal{L}\Psi \leq -A_3 c_1 \sum f_i \lambda_i^2 + (-A_1 + CA_2 + CA_3) \sum f_i - A_1 + CA_3$$

We now fix  $N$  such that (3.16) holds when  $|\lambda| > R$  while (3.20) holds when  $|\lambda| \leq R$ , for any  $s$  and  $\delta$  sufficiently small.

Case **(b)**  $|\nu_\mu - \nu_\lambda| \geq \beta$ . It follows from Lemma 2.2 that, for some  $\varepsilon > 0$ ,

$$\mathcal{L}(\underline{u} - u) \geq \sum f_i (\mu_i - \lambda_i) - (\underline{u} - u)_t \geq \varepsilon \left( 1 + \sum f_i \right)$$

By (3.15), we may fix  $s$  and  $\delta$  sufficiently small such that  $v \geq 0$  on  $\overline{M_T^\delta}$  and

$$(3.22) \quad \mathcal{L}v \leq -\frac{\varepsilon}{2} \left( 1 + \sum f_i \right) \text{ in } M_T^\delta.$$

Finally, we choose  $A_2$  large such that

$$(A_2 - K)\rho^2 \geq A_3 \sum_{l < n} |\nabla_l(u - \varphi)|^2 \text{ on } \partial M_T^\delta,$$

and then fix  $A_1$  sufficiently large so that (3.4) holds; in case **(a)** this follows from (3.17) when  $|\lambda| > R$ , and from (3.21) when  $|\lambda| \leq R$ , while in case **(b)** we obtain (3.4) from (3.6), (3.7), (3.22) and the following inequality

$$(3.23) \quad \sum f_i |\lambda_i| \leq \epsilon \sum_{i \neq r} f_i \lambda_i^2 + \frac{C}{\epsilon} \sum f_i + C$$

for any  $\epsilon > 0$  and index  $r$ , which is a consequence of (1.2), (1.3) and (1.13). For the proof of (3.23), we consider two cases. If  $\lambda_r < 0$  then, by (1.13)

$$\begin{aligned} \sum f_i |\lambda_i| &= 2 \sum_{\lambda_i > 0} f_i \lambda_i - \sum f_i \lambda_i \\ &\leq \epsilon \sum_{\lambda_i > 0} f_i \lambda_i^2 + \frac{1}{\epsilon} \sum_{\lambda_i > 0} f_i + K_1 \sum f_i + K_1. \end{aligned}$$

If  $\lambda_r > 0$ , we have by the concavity of  $f$ ,

$$\begin{aligned} \sum f_i |\lambda_i| &= \sum f_i \lambda_i - 2 \sum_{\lambda_i < 0} f_i \lambda_i \\ &\leq \epsilon \sum_{\lambda_i < 0} f_i \lambda_i^2 + \frac{1}{\epsilon} \sum_{\lambda_i < 0} f_i + \sum f_i + f(\lambda) - f(\mathbf{1}). \end{aligned}$$

This proves (3.23).  $\square$

Applying Lemma 3.1, by (3.5) we immediately derive a bound for the mixed tangential-normal derivatives at any point  $(x_0, t_0) \in \partial M_T$ ,

$$(3.24) \quad |\nabla_{n\alpha} u(x_0, t_0)| \leq C, \quad \forall \alpha < n$$

It remains to establish the double normal derivative estimate

$$(3.25) \quad |\nabla_{nn} u(x_0, t_0)| \leq C.$$

As in [6] and [7] we use an idea originally due to Trudinger [16].

For  $(x, t) \in \partial_s M_T$ , let  $\tilde{U}(x, t)$  be the restriction to  $T_x \partial M$  of  $U(x, t)$ , viewed as a bilinear map on the tangent space of  $M$  at  $x$ , and let  $\lambda'(\tilde{U})$  denote the eigenvalues of  $\tilde{U}$  with respect to the induced metric on  $\partial M$ . We show next that there are uniform positive constants  $c_0, R_0$  such that, for all  $R > R_0$ ,  $(\lambda'(\tilde{U}(x, t)), R) \in \Gamma$  and

$$(3.26) \quad f(\lambda'(\tilde{U}(x, t)), R) \geq f(\lambda(U(x, t))) + c_0, \quad \forall 0 \leq t \leq T, \quad \forall x \in \partial M.$$

It is known that (3.26) implies (3.25); see e.g. [6].

For  $R > 0$  sufficiently large, let

$$\begin{aligned} m_R &:= \min_{\partial_s M_T} [f(\lambda'(\tilde{U}), R) - f(\lambda(U))], \\ c_R &:= \min_{\partial_s M_T} [f(\lambda'(\underline{U}), R) - f(\lambda(\underline{U}))]. \end{aligned}$$



Note that  $(\lambda'(\tilde{U}(x, t)), R) \in \Gamma$  and  $(\lambda'(\tilde{U}(x, t)), R) \in \Gamma$  for all  $(x, t) \in \partial_s M_T$  for all  $R$  large, and it is clear that both  $m_R$  and  $c_R$  are increasing in  $R$ . We wish to show that for some uniform  $c_0 > 0$ ,

$$\tilde{m} := \lim_{R \rightarrow \infty} m_R \geq c_0.$$

Assume  $\tilde{m} < \infty$  (otherwise we are done) and fix  $R > 0$  such that  $c_R > 0$  and  $m_R \geq \frac{\tilde{m}}{2}$ . Let  $(x_0, t_0) \in \partial_s M_T$  such that  $m_R = f(\lambda'(\tilde{U}(x_0, t_0)), R)$ . Choose local orthonormal frames  $e_1, \dots, e_n$  around  $x_0$  as before such that  $e_n$  is the interior normal to  $\partial M$  along the boundary and  $U_{\alpha\beta}(x_0, t_0)$  ( $1 \leq \alpha, \beta \leq n-1$ ) is diagonal. Since  $u - \underline{u} = 0$  on  $\partial_s M_T$ , we have

$$(3.27) \quad U_{\alpha\beta} - \underline{U}_{\alpha\beta} = -\nabla_n(u - \underline{u})\sigma_{\alpha\beta} \quad \text{on } \partial_s M_T.$$

where  $\sigma_{\alpha\beta} = \langle \nabla_\alpha e_\beta, e_n \rangle$ . Similarly,

$$(3.28) \quad U_{\alpha\beta} - \nabla_{\alpha\beta}\varphi - \chi_{\alpha\beta}\varphi = -\nabla_n(u - \varphi)\sigma_{\alpha\beta} \quad \text{on } \partial_s M_T.$$

For an  $(n-1) \times (n-1)$  symmetric matrix  $\{r_{\alpha,\beta}\}$  with  $(\lambda'(\{r_{\alpha,\beta}\}), R) \in \Gamma$ , define

$$\tilde{F}[r_{\alpha\beta}] := f(\lambda'(\{r_{\alpha,\beta}\}), R)$$

and

$$\tilde{F}_0^{\alpha\beta} = \frac{\partial \tilde{F}}{\partial r_{\alpha\beta}}[U_{\alpha\beta}(x_0, t_0)].$$

We see that  $\tilde{F}$  is concave since so is  $f$ , and therefore by (3.27),

$$\nabla_n(u - \underline{u})(x_0, t_0)\tilde{F}_0^{\alpha\beta}\sigma_{\alpha\beta}(x_0) \geq \tilde{F}[\underline{U}_{\alpha\beta}(x_0, t_0)] - \tilde{F}[U_{\alpha\beta}(x_0, t_0)] \geq c_R - m_R.$$

Suppose that

$$\nabla_n(u - \underline{u})(x_0, t_0)\tilde{F}_0^{\alpha\beta}\sigma_{\alpha\beta}(x_0) \leq \frac{c_R}{2}$$

then  $m_R \geq c_R/2$  and we are done. So we shall assume

$$\nabla_n(u - \underline{u})(x_0, t_0)\tilde{F}_0^{\alpha\beta}\sigma_{\alpha\beta}(x_0) > \frac{c_R}{2}.$$

Consequently,

$$(3.29) \quad \tilde{F}_0^{\alpha\beta}\sigma_{\alpha\beta}(x_0) \geq \frac{c_R}{2\nabla_n(u - \underline{u})(x_0, t_0)} \geq 2\epsilon_1 c_R$$

for some constant  $\epsilon_1 > 0$  depending on  $\max_{\partial_s M_T} |\nabla u|$ . By continuity we may assume  $\eta := \tilde{F}_0^{\alpha\beta}\sigma_{\alpha\beta} \geq \epsilon_1 c_R$  on  $\overline{M_T^\delta}$  by requiring  $\delta$  small (which may depend on the fixed  $R$ ). Define in  $M_T^\delta$ ,

$$(3.30) \quad \Phi = -\nabla_n(u - \varphi) + \frac{Q}{\eta}$$

where

$$Q = \tilde{F}_0^{\alpha\beta}(\nabla_{\alpha\beta}\varphi + \chi_{\alpha\beta} - U_{\alpha\beta}(x_0, t_0)) - \underline{u}_t - \psi + u_t(x_0, t_0) + \psi(x_0, t_0)$$

is smooth in  $M_T^\delta$ . By (3.5) we have

$$(3.31) \quad \mathcal{L}\Phi \leq -\mathcal{L}\nabla_n u + C\left(1 + \sum F^{ii}\right) \leq C\left(1 + \sum f_i|\lambda_i| + \sum f_i\right).$$

From (3.28) we see that  $\Phi(x_0, t_0) = 0$  and

$$(3.32) \quad \Phi \geq 0 \text{ on } \overline{M_T^\delta} \cap \partial_s M_T,$$

since for  $(x, t) \in \partial_s M_T$ , by the concavity of  $\tilde{F}$ ,

$$\begin{aligned} \tilde{F}_0^{\alpha\beta}(U_{\alpha\beta}(x, t) - U_{\alpha\beta}(x_0, t_0)) &\geq \tilde{F}(\tilde{U}(x, t)) - \tilde{F}(\tilde{U}(x_0, t_0)) \\ &= \tilde{F}(\tilde{U}(x, t)) - m_R - u_t(x_0, t_0) - \psi(x_0, t_0) \\ &\geq \psi(x, t) + u_t(x, t) - u_t(x_0, t_0) - \psi(x_0, t_0). \end{aligned}$$

On the other hand, on  $\partial_b M_T^\delta$  we have  $\nabla_n(u - \varphi) = 0$  and therefore, by (3.32),

$$(3.33) \quad \Phi(x, 0) \geq \Phi(\hat{x}, 0) - Cd(x) \geq -Cd(x),$$

where  $C$  depends on  $C^1$  bounds of  $\nabla^2\varphi(\cdot, 0)$ ,  $\underline{u}_t(\cdot, 0)$ ,  $\psi(\cdot, 0)$  on  $\bar{M}$ , and  $\hat{x} \in \partial M$  satisfies  $d(x) = \text{dist}(x, \hat{x})$  for  $x \in M$ ; when  $d(x)$  is sufficiently small,  $\hat{x}$  is unique.

Finally, note that  $|\Phi| \leq C$  in  $M_T^\delta$ . So we may apply Lemma 3.1 to derive  $\Psi + \Phi \geq 0$  on  $\partial M_T^\delta$  and

$$(3.34) \quad \mathcal{L}(\Psi + \Phi) \leq 0 \text{ in } M_T^\delta$$

for  $A_1, A_2, A_3$  sufficiently large. By the maximum principle,  $\Psi + \Phi \geq 0$  in  $M_T^\delta$ . This gives  $\nabla_n \Phi(x_0, t_0) \geq -\nabla_n \Psi(x_0, t_0) \geq -C$  since  $\Phi + \Psi = 0$  at  $(x_0, t_0)$ , and therefore,  $\nabla_{nn} u(x_0, t_0) \leq C$ .

Consequently, we have obtained *a priori* bounds for all second derivatives of  $u$  at  $(x_0, t_0)$ . It follows that  $\lambda(U(x_0, t_0))$  is contained in a compact subset (independent of  $u$ ) of  $\Gamma$  by assumptions (1.4). Therefore,

$$c_0 \equiv \frac{f(\lambda(U(x_0, t_0)) + R\mathbf{e}_n) - f(\lambda(U(x_0, t_0)))}{2} > 0$$

where  $\mathbf{e}_n = (0, \dots, 0, 1) \in \mathbb{R}^n$ . By Lemma 1.2 in [1] we have

$$\tilde{m} \geq m_{R'} \geq f(\lambda(U(x_0, t_0)) + R'\mathbf{e}_n) - c_0 - f(\lambda(U(x_0, t_0))) \geq c_0$$

for  $R' \geq R$  sufficiently large. The proof of (1.14) in Theorem 1.1 is complete.

*Remark 3.2.* When  $M$  is a bounded smooth in  $\mathbb{R}^n$ , one can make use of an identity in [1], and modify the operator  $\mathcal{L}$ , to derive the boundary estimates without using assumption (1.13). We omit the proof here since it is similar to the elliptic case in [7] which we refer the reader to for details.

4. EXISTENCE AND  $C^1$  ESTIMATES

In order to prove Theorem 1.9 it remains to derive the  $C^1$  estimate

$$(4.1) \quad |u|_{C^0(\bar{M}_T)} + \max_{\bar{M} \times [t_0, T]} (|\nabla u| + |u_t|) \leq C$$

for any  $t_0 \in (0, T)$ , where  $C$  may depend on  $t_0$ . Indeed, by assumption (1.4) we see that equation (1.1) becomes uniformly parabolic once the  $C^{2,1}$  estimate

$$|u|_{C^{2,1}(\bar{M} \times [t_0, T])} \leq C$$

is established, which yields  $|u|_{C^{2+\alpha, 1+\alpha/2}(\bar{M} \times [t_0, T])} \leq C$  by Evans-Krylov theorem [3, 13] (see e.g. [15]). Higher order estimates now follow from the classical Schauder theory of linear parabolic equations, and one obtains a smooth admissible solution in  $0 \leq t \leq T$  by the short time existence and continuation. We refer the reader to [15] for details.

Let  $h \in C^2(\bar{M}_T)$  be the solution of  $\Delta h + \text{tr} \chi = 0$  in  $\bar{M}_T$  with  $h = \varphi$  on  $\partial M_T$ . By the maximum principle we have  $\underline{u} \leq u \leq h$  which gives a bound

$$(4.2) \quad |u|_{C^0(\bar{M}_T)} + \max_{\partial M_T} |\nabla u| \leq C.$$

For the bound of  $u_t$  we have the following maximum principle.

**Lemma 4.1.**

$$(4.3) \quad |u_t(x, t)| \leq \max_{\partial M_T} |u_t| + t \sup_{M_T} |\psi_t|, \quad \forall (x, t) \in \bar{M}_T$$

Moreover, if there is a convex function in  $C^2(\bar{M})$  then

$$(4.4) \quad \sup_{M_T} |u_t| \leq \max_{\partial M_T} |u_t| + C \sup_{M_T} |\nabla^2 \psi|$$

where  $C$  is independent of  $T$ .

*Proof.* We have the following identities:  $\mathcal{L}u_t = \psi_t$  and

$$|\mathcal{L}(u_t + \psi)| = |F^{ij} \nabla_{ij} \psi| \leq |\nabla^2 \psi| \sum F^{ii}.$$

So Lemma 4.1 is an immediate consequence of the maximum principle.  $\square$

It remains to derive the gradient estimate

$$(4.5) \quad \sup_{M_T} |\nabla u|^2 \leq C \left( |u|_{C^0(\bar{M}_T)} + \sup_{\partial M_T} |\nabla u|^2 \right)$$

in each of the cases (i)-(iv) in Theorem 1.9. We shall omit case (i) which is trivial, and consider cases (ii)-(iv) following ideas from [14, 17, 6] in the elliptic case.

Let  $\phi$  be a function to be chosen and assume that  $|\nabla u|e^\phi$  achieves a maximum at an interior point  $(x_0, t_0) \in M_T$ . As before we choose local orthonormal frames at  $x_0$  such that both  $U_{ij}$  and  $F^{ij}$  are diagonal at  $(x_0, t_0)$  where

$$(4.6) \quad \frac{\nabla_k u \nabla_k u_t}{|\nabla u|^2} + \phi_t \geq 0, \quad \frac{\nabla_k u \nabla_{ik} u}{|\nabla u|^2} + \nabla_i \phi = 0, \quad \forall i = 1, \dots, n,$$

$$(4.7) \quad F^{ii} \frac{\nabla_k u \nabla_{iik} u + \nabla_{ik} u \nabla_{ik} u}{|\nabla u|^2} - 2F^{ii} \frac{(\nabla_k u \nabla_{ik} u)^2}{|\nabla u|^4} + F^{ii} \nabla_{ii} \phi \leq 0.$$

We have for any  $0 < \epsilon < 1$ ,

$$\sum_k (\nabla_{ik} u)^2 = \sum_k (U_{ik} - \chi_{ik})^2 \geq (1 - \epsilon) U_{ii}^2 - \frac{C}{\epsilon}.$$

and

$$\left( \sum_k \nabla_k u \nabla_{ik} u \right)^2 \leq (1 + \epsilon) |\nabla_i u|^2 U_{ii}^2 + \frac{C}{\epsilon} |\nabla u|^2.$$

Let  $\epsilon = \frac{1}{3}$  and  $J = \{i : 2(n+2)|\nabla_i u|^2 > |\nabla u|^2\}$ ; note that  $J \neq \emptyset$ . By (4.6) and (4.7) we obtain

$$(4.8) \quad \begin{aligned} & \frac{1}{3} F^{ii} U_{ii}^2 - 2|\nabla u|^2 \sum_{i \in J} F^{ii} |\nabla_i \phi|^2 + |\nabla u|^2 (F^{ii} \nabla_{ii} \phi - \phi_t) \\ & \leq C(1 - K_0 |\nabla u|^2) \sum F^{ii} + C|\nabla u| \end{aligned}$$

where  $K_0 = \inf_{k,l} R_{klkl}$ .

Let

$$\phi = -\log(1 - bv^2) + A(\underline{u} + w - Bt)$$

where  $v$  is a positive function,  $A$ ,  $B$  and  $b$  are constant, all to be determined;  $b$  will be chosen sufficiently small such that  $14bv^2 \leq 1$  in  $\overline{M_T}$ , while  $A = 0$  in cases (ii) and (iii). By straightforward calculations,

$$\nabla_i \phi = \frac{2bv \nabla_i v}{1 - bv^2} + A \nabla_i (\underline{u} + w), \quad \phi_t = \frac{2bv v_t}{1 - bv^2} + A(\underline{u}_t - B)$$

and

$$\begin{aligned} \nabla_{ii} \phi &= \frac{2bv \nabla_{ii} v + 2b |\nabla_i v|^2}{1 - bv^2} + \frac{4b^2 v^2 |\nabla_i v|^2}{(1 - bv^2)^2} + A \nabla_{ii} (\underline{u} + w) \\ &= \frac{2bv \nabla_{ii} v}{1 - bv^2} + \frac{2b(1 + bv^2) |\nabla_i v|^2}{(1 - bv^2)^2} + A \nabla_{ii} (\underline{u} + w). \end{aligned}$$

Plugging these into (4.8), we obtain

$$(4.9) \quad \begin{aligned} & \frac{1}{3} F^{ii} U_{ii}^2 + |\nabla u|^2 \sum_{i \in J} F^{ii} \left( \frac{b(1 - 7bv^2) |\nabla_i v|^2}{(n+2)(1 - bv^2)^2} - CA^2 \right) \\ & + \frac{2bv |\nabla u|^2}{1 - bv^2} (F^{ii} \nabla_{ii} v - v_t) + A |\nabla u|^2 (F^{ii} \nabla_{ii} (\underline{u} + w) - \underline{u}_t + B) \\ & \leq C(1 - K_0 |\nabla u|^2) \sum F^{ii} + C|\nabla u|. \end{aligned}$$

In both cases (ii) and (iv) we take

$$v = \underline{u} - u + \sup_{\bar{M}_T} (u - \underline{u}) + 1 \geq 1.$$

Let  $\mu = \lambda(\nabla^2 \underline{u}(x_0, t_0) + \chi(x_0))$ ,  $\lambda = \lambda(\nabla^2 u(x_0, t_0) + \chi(x_0))$  and  $\beta$  as in (2.24). Suppose first that  $|\nu_\mu - \nu_\lambda| \geq \beta$ . By Lemma 2.2 and the assumptions that  $\sum f_i \lambda_i \geq 0$  and  $\nabla^2 w \geq \chi$  we see that,

$$F^{ii} \nabla_{ii}(\underline{u} + w) - \underline{u}_t + B \geq F^{ii} \nabla_{ii} v - v_t + (B - u_t) \geq \varepsilon \sum F^{ii} + \varepsilon + (B - u_t)$$

for some  $\varepsilon > 0$ . Let  $A = A_1 K_0^- / \varepsilon$ ,  $K_0^- = \max\{-K_0, 0\}$  and fix  $A_1, B$  sufficiently large. A bound  $|\nabla u| \leq C$  follows from (4.9) in both cases (ii) and (iv).

We now consider the case  $|\nu_\mu - \nu_\lambda| < \beta$ . By (2.26) and (4.9) we see that if  $|\nabla u|$  is sufficiently large,

$$(4.10) \quad \begin{aligned} \frac{\beta}{\sqrt{n}}(|\lambda|^2 + c_1 |\nabla u|^4) \sum F^{ii} &\leq F^{ii} U_{ii}^2 + 2c_1 |\nabla u|^4 \sum_{i \in J} F^{ii} \\ &\leq C(1 - K_0 |\nabla u|^2) \sum F^{ii} + C |\nabla u| \end{aligned}$$

where  $c_1 > 0$ .

Suppose  $|\lambda| \geq R$  for  $R$  sufficiently large. Then

$$(4.11) \quad \frac{\beta}{\sqrt{n}}(|\lambda|^2 + c_1 |\nabla u|^4) \sum F^{ii} \geq \frac{2\beta |\lambda| \sqrt{c_1}}{\sqrt{n}} |\nabla u|^2 \sum F^{ii} \geq c_2 |\nabla u|^2$$

for some uniform  $c_2 > 0$ . We obtain from (4.10) and (4.11) a bound for  $|\nabla u(x_0, t_0)|$ .

Suppose now that  $|\lambda| \leq R$ . Then  $\sum F^{ii}$  has a positive lower bound by (3.18) and (3.19). Therefore a bound  $|\nabla u(x_0, t_0)|$  follows from (4.10) again. This completes the proof of (4.5) in cases (ii) and (iv).

For case (iii) we choose  $A = 0$  and

$$(4.12) \quad \phi = (u - \inf_{\overline{M_T}} u + 1)^2.$$

By (4.9)

$$(4.13) \quad |\nabla u|^4 \sum_{i \in J} F^{ii} \leq C(1 - K_0 |\nabla u|^2) \sum F^{ii} + C |\nabla u|.$$

By (4.6) we see that  $U_{ii} \leq 0$  for each  $i \in J$  if  $|\nabla u|$  is sufficiently large, and a bound for  $|\nabla u(x_0, t_0)|$  therefore follows from (4.13) and assumption (1.20).

## 5. APPENDIX: PROOF OF LEMMA 2.5

In this Appendix we present a proof of Lemma 2.5 (Theorem 5.10) for the reader's convenience. The basic ideas of the proof are adopted from [6].

For  $\sigma \in \mathbb{R}$  define

$$\Sigma^\sigma := \{(\lambda, p) \in \Gamma \times \mathbb{R} : f(\lambda) - p > \sigma\}.$$

Let  $\partial\Sigma^\sigma$  be the boundary of  $\Sigma^\sigma$  and  $T_{\hat{\lambda}}\partial\Sigma^\sigma$  denote the tangent hyperplane to  $\partial\Sigma^\sigma$  at  $\hat{\lambda} \in \partial\Sigma^\sigma$ . The unit normal vector to  $\partial\Sigma^\sigma$  at  $\hat{\lambda}$  is given by

$$\nu_{\hat{\lambda}} = \frac{(Df(\lambda), -1)}{\sqrt{1 + |Df(\lambda)|^2}}.$$

In addition, for  $\hat{\mu} \in \Gamma \times \mathbb{R}$  let

$$\begin{aligned} \hat{S}_\mu^\sigma &:= \{\hat{\lambda} \in \partial\Sigma^\sigma : (\hat{\mu} - \hat{\lambda}) \cdot \nu_{\hat{\lambda}} \leq 0\}, \\ \mathcal{B}_\sigma^+ &:= \{\hat{\mu} \in \Gamma \times \mathbb{R} : \hat{S}_\mu^\sigma \cap \Gamma \times \{a\} \text{ is compact, } \forall a \in \mathbb{R}\}, \\ V^\sigma &:= \mathcal{B}_\sigma^+ \setminus \Sigma^\sigma, \end{aligned}$$

and for  $\hat{\mu} \in V^\sigma$ ,

$$\mathcal{B}_\sigma^+(\hat{\mu}) = \{t\hat{\lambda} + (1-t)\hat{\mu} : \hat{\lambda} \in \hat{S}_\mu^\sigma, 0 \leq t \leq 1\}.$$

For convenience we shall write  $\hat{\lambda} = (\lambda, p)$ ,  $\hat{\mu} = (\mu, q)$  and  $\tilde{f}(\hat{\lambda}) = f(\lambda) - p$  in this section.

**Lemma 5.1.** *Let  $\delta > 0$ ,  $\hat{\mu} \in V^\sigma$ . Then*

$$H_{\hat{\mu}}(R) := \min_{\hat{\lambda} \in \partial\Sigma^\sigma \cap \{|\lambda|=R, |p-q| \leq \delta\}} (\hat{\mu} - \hat{\lambda}) \cdot \nu_{\hat{\lambda}} > 0$$

for

$$R > R_{\hat{\mu}} := \max_{(\lambda, p) \in \hat{S}_\mu^\sigma \cap \Gamma \times [q-\delta, q+\delta]} |\lambda|.$$

**Lemma 5.2.** *Let  $\hat{\mu} \in V^\sigma$ . Then  $\mathcal{B}_\sigma^+(\hat{\mu}) \subset V^\sigma$  and  $\mathcal{B}_\sigma^+(\hat{\mu}') \subset \mathcal{B}_\sigma^+(\hat{\mu})$  for  $\hat{\mu}' \in \mathcal{B}_\sigma^+(\hat{\mu})$ .*

*Proof.* Let  $\hat{\mu}_t = t\hat{\lambda} + (1-t)\hat{\mu}$  for  $t \in [0, 1]$  and  $\hat{\lambda} \in \hat{S}_\mu^\sigma$ . Since  $\partial\Sigma^\sigma$  is convex,

$$\begin{aligned} (\hat{\mu}_t - \hat{\zeta}) \cdot \nu_{\hat{\zeta}} &= (1-t)(\hat{\mu} - \hat{\zeta}) \cdot \nu_{\hat{\zeta}} + t(\hat{\lambda} - \hat{\zeta}) \cdot \nu_{\hat{\zeta}} \\ &> t(\hat{\lambda} - \hat{\zeta}) \cdot \nu_{\hat{\zeta}} \geq 0, \quad \forall \hat{\zeta} \in \partial\Sigma^\sigma \setminus \hat{S}_\mu^\sigma. \end{aligned}$$

This shows  $\hat{S}_{\hat{\mu}_t}^\sigma \subset \hat{S}_\mu^\sigma$  and therefore  $\hat{\mu}_t \in V^\sigma$ . Clearly  $\mathcal{B}_\sigma^+(\hat{\mu}) \subset \mathcal{B}_\sigma^+(\hat{\mu})$ .  $\square$

**Lemma 5.3.** *The cone  $\mathcal{B}_\sigma^+$  is open.*

*Proof.* Let  $\hat{\mu} \in V^\sigma$  and  $a \in \mathbb{R}$ . Since  $\hat{S}_\mu^\sigma \cap \Gamma \times \{a\}$  is compact,  $\hat{S}_\mu^\sigma \cap \Gamma \times \{a\} \subset B_R \times \{a\}$  for sufficiently large  $R$ . Therefore

$$\alpha := \frac{1}{2\sqrt{n}} \min_{\hat{\zeta} \in \partial\Sigma^\sigma \cap \partial B_R \times \{a\}} (\hat{\mu} - \hat{\zeta}) \cdot \nu_{\hat{\zeta}} > 0$$

and

$$(\hat{\mu} - \alpha \hat{\mathbf{1}} - \hat{\lambda}) \cdot \nu_{\hat{\lambda}} \geq -\sqrt{n}\alpha + (\hat{\mu} - \hat{\lambda}) \cdot \nu_{\hat{\lambda}} \geq \sqrt{n}\alpha > 0, \quad \forall \hat{\lambda} \in \partial\Sigma^\sigma \cap \partial B_R \times \{a\}$$

where  $\hat{\mathbf{1}} = (\mathbf{1}, 0) \in \mathbb{R}^{n+1}$ . This proves that  $\hat{S}_{(\hat{\mu}-\alpha\hat{\mathbf{1}})}^\sigma \cap \Gamma \times \{a\} \subset \partial\Sigma^\sigma \cap B_R \times \{a\}$  and hence  $\hat{\mu} - \alpha\hat{\mathbf{1}} \in V^\sigma$ . On the other hand, for any  $\hat{\lambda} \in \hat{S}_{\hat{\mu}}^\sigma \cap \Gamma \times \{a\}$ ,

$$(\hat{\mu} - \alpha\hat{\mathbf{1}} - \hat{\lambda}) \cdot \nu_{\hat{\lambda}} \leq -\alpha\hat{\mathbf{1}} \cdot \nu_{\hat{\lambda}} = -\frac{\alpha \sum f_i(\lambda)}{\sqrt{1 + |Df|^2}} < 0.$$

So  $\mathcal{B}_\sigma^+(\hat{\mu}) \subset \mathcal{B}_\sigma^+(\hat{\mu} - \alpha\hat{\mathbf{1}}) \subset V^\sigma$ . Clearly  $\mathcal{B}_\sigma^+(\hat{\mu} - \alpha\hat{\mathbf{1}})$  contains a ball centered at  $\hat{\mu}$ .  $\square$

**Lemma 5.4.** *Let  $K$  be a compact subset of  $V^\sigma = \mathcal{B}_\sigma^+ \setminus \Sigma^\sigma$ . Then, for any  $\delta > 0$ ,*

$$\sup_{\hat{\mu} \in K} \max_{\hat{\lambda} \in \hat{S}_{\hat{\mu}}^\sigma \cap \Gamma \times [a-\delta, b+\delta]} |\lambda| < \infty.$$

where  $a = \min\{q | \hat{\mu} \in K\}$ ,  $b = \max\{q | \hat{\mu} \in K\}$ .

*Proof.* Suppose this is not true. Then for each integer  $k \geq 1$  there exists  $\hat{\mu}_k \in K$  and  $\hat{\lambda}_k \in \hat{S}_{\hat{\mu}_k}^\sigma \cap \Gamma \times [a-\delta, b+\delta]$  with  $|\lambda_k| \geq k$ . By the compactness of  $K$  we may assume  $\hat{\mu}_k \rightarrow \hat{\mu} \in K$  as  $k \rightarrow \infty$ . Thus

$$\limsup_{k \rightarrow \infty} (\hat{\mu} - \hat{\lambda}_k) \cdot \nu_{\hat{\lambda}_k} = \limsup_{k \rightarrow \infty} (\hat{\mu} - \hat{\mu}_k) \cdot \nu_{\hat{\lambda}_k} + \limsup_{k \rightarrow \infty} (\hat{\mu}_k - \hat{\lambda}_k) \cdot \nu_{\hat{\lambda}_k} \leq 0.$$

On the other hand, by Lemma 5.1 there exists  $\epsilon > 0$  such that

$$(\hat{\mu} - \hat{\lambda}_k) \cdot \nu_{\hat{\lambda}_k} \geq \epsilon, \quad \forall k > \max_{\hat{\lambda} \in \hat{S}_{\hat{\mu}}^\sigma \cap \Gamma \times [q-\delta, q+\delta]} |\lambda|.$$

This is a contradiction.  $\square$

Let  $\hat{\mu} \in \overline{\Sigma^\sigma}$  and  $\hat{\lambda} \in \partial\Sigma^\sigma$ . By the convexity of  $\partial\Sigma^\sigma$ , the open segment

$$(\hat{\mu}, \hat{\lambda}) := \{t\hat{\mu} + (1-t)\hat{\lambda} : 0 < t < 1\}$$

is completely contained in either  $\partial\Sigma^\sigma$  or  $\Sigma^\sigma$  by condition (1.2). Therefore,

$$\tilde{f}(t\hat{\mu} + (1-t)\hat{\lambda}) > \sigma, \quad \forall 0 < t < 1$$

unless  $(\hat{\mu}, \hat{\lambda}) \subset \partial\Sigma^\sigma$  which is equivalent to  $\hat{S}_{\hat{\mu}}^\sigma = \hat{S}_{\hat{\lambda}}^\sigma$ .

For  $\hat{\mu} \in \overline{\Sigma^\sigma}$ ,  $\delta > 0$  and  $R > |\mu|$  let

$$\Theta_R(\hat{\mu}) := \inf_{\hat{\lambda} \in \{|\lambda|=R, |p-q| \leq \delta\} \cap \partial\Sigma^\sigma} \max_{0 \leq t \leq 1} \tilde{f}(t\hat{\mu} + (1-t)\hat{\lambda}) - \sigma \geq 0.$$

Clearly  $\Theta_R(\hat{\mu}) = 0$  if and only if  $(\hat{\mu}, \hat{\lambda}) \subset \partial\Sigma^\sigma$  for some  $\hat{\lambda} \in \{|\lambda| = R, |p-q| \leq \delta\} \cap \partial\Sigma^\sigma$ , since the set  $\{|\lambda| = R, |p-q| \leq \delta\} \cap \partial\Sigma^\sigma$  is compact.

**Lemma 5.5.** *For  $\hat{\mu} \in \overline{\Sigma^\sigma}$ ,  $\Theta_R(\hat{\mu})$  is nondecreasing in  $R$ . Moreover, if  $\Theta_{R_0}(\hat{\mu}) > 0$  for some  $R_0 \geq |\mu|$  then  $\Theta_{R'}(\hat{\mu}) > \Theta_R(\hat{\mu})$  for all  $R' > R \geq R_0$ .*

*Proof.* We shall write  $\Theta_R = \Theta_R(\hat{\mu})$  when there is no possible confusion. Suppose  $\Theta_{R_0}(\hat{\mu}) > 0$  for some  $R_0 \geq |\mu|$ . Let  $R' > R \geq R_0$  and assume that  $\Theta_{R'}$  is achieved at  $\hat{\lambda}_{R'} \in \{\hat{\lambda} \in \Gamma \times \mathbb{R} \mid |\lambda| = R', |p - q| \leq \delta\} \cap \partial\Sigma^\sigma$ , that is,

$$\Theta_{R'} = \max_{0 \leq t \leq 1} \tilde{f}(t\hat{\mu} + (1-t)\hat{\lambda}_{R'}) - \sigma.$$

Let  $P$  be the (two dimensional) plane through  $\hat{\mu}, \hat{\lambda}_{R'}$  and the point of  $(0, q)$ . There is a point  $\hat{\lambda}_R \in \{|\lambda| = R, |p - q| \leq \delta\}$  which lies on the curve  $P \cap \Sigma^\sigma$ . Note that  $\hat{\mu}, \hat{\lambda}_R$  and  $\hat{\lambda}_{R'}$  are not on a straight line, for  $(\hat{\mu}, \hat{\lambda}_R)$  can not be part of  $(\hat{\mu}, \hat{\lambda}_{R'})$  since  $\Theta_{R_0} > 0$  and  $\partial\Sigma^\sigma$  is convex. We see that

$$\max_{0 \leq t \leq 1} \tilde{f}(t\hat{\mu} + (1-t)\hat{\lambda}_R) - \sigma < \Theta_{R'}$$

by condition (1.2). This proves  $\Theta_R < \Theta_{R'}$ .  $\square$

**Lemma 5.6.** *Let  $\hat{\mu} \in \partial\Sigma^\sigma \cap \mathcal{B}_\sigma^+$  and  $\delta > 0$ . Then*

$$\Theta_R(\hat{\mu}) > 0, \forall R > \max_{\hat{\lambda} \in \hat{S}_\mu^\sigma \cap \Gamma \times [q-\delta, q+\delta]} |\lambda|.$$

*Proof.* This is obvious.  $\square$

**Lemma 5.7.** *For  $\hat{\mu} \in \partial\Sigma^\sigma \cap \mathcal{B}_\sigma^+$  and  $\delta > 0$ , let  $N_{\hat{\mu}} = 2 \max_{\hat{\lambda} \in \hat{S}_\mu^\sigma \cap \Gamma \times [q-\delta, q+\delta]} |\lambda|$ . Then for any  $\hat{\lambda} \in \partial\Sigma^\sigma$  with  $|p - q| \leq \delta$ , when  $|\lambda| \geq N_{\hat{\mu}}$ ,*

$$(5.1) \quad \sum f_i(\lambda)(\mu_i - \lambda_i) - (q - p) \geq \Theta_{N_{\hat{\mu}}}(\hat{\mu}) > 0.$$

*Proof.* By the concavity of  $\tilde{f}$  with respect to  $\hat{\lambda}$ ,

$$\begin{aligned} & \sum f_i(\lambda)(\mu_i - \lambda_i) - (q - p) \\ & \geq \max_{0 \leq t \leq 1} \tilde{f}(t\hat{\mu} + (1-t)\hat{\lambda}) - \sigma, \quad \forall \hat{\mu}, \hat{\lambda} \in \partial\Sigma^\sigma. \end{aligned}$$

So Lemma 5.7 follows from Lemma 5.5 and Lemma 5.6.  $\square$

**Lemma 5.8.** *Let  $K$  be a compact subset of  $\partial\Sigma^\sigma \cap \mathcal{B}_\sigma^+$  and  $\eta > 0$ . Define*

$$\begin{aligned} a &= \min\{q \mid \hat{\mu} \in K\}, \quad b = \max\{q \mid \hat{\mu} \in K\}, \quad \delta = |b - a| + \eta \\ N_K &:= \sup_{\hat{\mu} \in K} N_{\hat{\mu}}, \quad \Theta_K := \inf_{\hat{\mu} \in K} \Theta_{N_K} \hat{\mu}. \end{aligned}$$

*Then for any  $\hat{\lambda} \in \partial\Sigma^\sigma \cap \Gamma \times [a - \eta, b + \eta]$ , when  $|\lambda| \geq N_K$ ,*

$$(5.2) \quad \sum f_i(\lambda)(\mu_i - \lambda_i) - (q - p) \geq \Theta_K > 0, \quad \forall \hat{\mu} \in K.$$

*Proof.* From Lemma 5.4 we see that  $N_K < \infty$  and consequently,

$$\Theta_K := \inf_{\hat{\mu} \in K} \inf_{\hat{\lambda} \in \{\lambda \mid |\lambda| = N_K, |p - q| \leq \delta\} \cap \partial\Sigma^\sigma} \max_{0 \leq t \leq 1} \tilde{f}(t\hat{\mu} + (1-t)\hat{\lambda}) - \sigma > 0$$

by the continuity of  $\tilde{f}$ . Now (5.2) follows from Lemma 5.7.  $\square$



**Theorem 5.9.** *Let  $\delta > 0$ ,  $\hat{\mu} = (\mu, p) \in \mathcal{B}_\sigma^+$  and  $0 < \varepsilon < \frac{1}{2} \text{dist}(\hat{\mu}, \partial \mathcal{B}_\sigma^+)$ . There exist positive constants  $\theta_{\hat{\mu}}$ ,  $R_{\hat{\mu}}$  such that for any  $\hat{\lambda} = (\lambda, q) \in \partial \Sigma^\sigma$  with  $|p - q| \leq \delta$ , when  $|\lambda| \geq R_{\hat{\mu}}$ ,*

$$(5.3) \quad \sum f_i(\lambda)(\mu_i - \lambda_i) - (q - p) \geq \theta_{\hat{\mu}} + \varepsilon \sum f_i(\lambda).$$

*Proof.* We first note that  $\hat{\mu}^\varepsilon := \hat{\mu} - \varepsilon \hat{\mathbf{1}} \in \mathcal{B}_\sigma^+$ . If  $\hat{\mu}^\varepsilon \in \overline{\Sigma}^\sigma$  then  $\Theta_{R_0}(\hat{\mu}^\varepsilon) > 0$  for some  $R_0 > 0$ . By Lemma 5.7 we see (5.3) hold when  $|\lambda| \geq R_0$ .

Suppose now that  $\hat{\mu}^\varepsilon \in \mathcal{B}_\sigma^+ \setminus \overline{\Sigma}^\sigma$  and  $\hat{\lambda} \in \partial \Sigma^\sigma \setminus \hat{S}_{\hat{\mu}^\varepsilon}^\sigma$  with  $|p - q| \leq \delta$ . the segment  $(\hat{\mu}^\varepsilon, \hat{\lambda})$  goes through  $\hat{S}_{\hat{\mu}^\varepsilon}^\sigma \cap \Gamma \times [q - \delta, q + \delta]$  at a point  $\hat{\lambda}'$ . By the concavity of  $\tilde{f}$  and Lemma 5.8 applied to  $K = \hat{S}_{\hat{\mu}^\varepsilon}^\sigma \cap \Gamma \times [q - \delta, q + \delta]$ , we obtain

$$\sum \tilde{f}_i(\hat{\lambda})(\hat{\mu}_i^\varepsilon - \hat{\lambda}_i) \geq \sum \tilde{f}_i(\hat{\lambda})(\hat{\lambda}'_i - \hat{\lambda}_i) \geq \Theta_K > 0.$$

This proves (5.3) for  $\theta_{\hat{\mu}} = \min\{\Theta_{R_0}, \Theta_K\}$ ,  $R_{\hat{\mu}} = \max\{R_0, N_K\}$ .  $\square$

**Theorem 5.10.** *Let  $K$  be a compact subset of  $\Gamma \times \mathbb{R}$ ,  $\eta > 0$ , and let  $a, b, \delta$  be defined as in Lemma 5.8. Suppose that  $\hat{S}_{\hat{\mu}}^\sigma[a, b] := \hat{S}_{\hat{\mu}}^\sigma \cap \Gamma \times [a, b]$  is compact for any  $\hat{\mu} \in K$ . Then there exist  $\varepsilon, \theta_K, R_K > 0$  such that for any  $\hat{\lambda} \in \partial \Sigma^\sigma \cap \Gamma \times [a - \eta, b + \eta]$ , when  $|\lambda| \geq R_K$ ,*

$$(5.4) \quad \sum f_i(\lambda)(\mu_i - \lambda_i) - (q - p) \geq \theta_K + \varepsilon \sum f_i(\lambda), \quad \forall \hat{\mu} \in K.$$

Furthermore, for any closed interval  $[c, d]$ ,  $\theta_K$  and  $R_K$  can be chosen so that (5.4) holds uniformly in  $\sigma \in [c, d]$ .

*Proof.* Let  $K_1 = \{\hat{\mu} \in K : \hat{\mu}^{3\varepsilon/2} \in \Sigma^\sigma\}$ ,  $K_2 = \{\hat{\mu} \in K : \hat{\mu}^{3\varepsilon/2} \in V^\sigma\}$  and

$$W := \cup_{\hat{\mu} \in K_2} \hat{S}_{\hat{\mu}^{3\varepsilon/2}}^\sigma.$$

By the concavity of  $\tilde{f}$  and compactness on  $\overline{K_1}$  we have

$$(5.5) \quad \begin{aligned} \sum \tilde{f}_i(\hat{\lambda})(\hat{\mu}_i^\varepsilon - \hat{\lambda}_i) &\geq \tilde{f}(\hat{\mu}^\varepsilon) - \tilde{f}(\hat{\lambda}) \\ &\geq \min_{\hat{\zeta} \in \overline{K_1}} \tilde{f}(\hat{\zeta}^\varepsilon) - \sigma > 0, \quad \forall \hat{\mu} \in K_1, \hat{\lambda} \in \partial \Sigma^\sigma. \end{aligned}$$

Next, by Lemma 5.4,

$$R_0 := \sup_{(\zeta, r) \in W \cap \Gamma \times [a, b]} |\zeta| < \infty.$$

So  $\bar{W} \cap \Gamma \times [a, b]$  is a compact subset of  $\mathcal{B}_\sigma^+ \cap \partial \Sigma^\sigma$ . Applying Lemma 5.8 to  $\bar{W}$ , we obtain for any  $\hat{\lambda} \in \partial \Sigma^\sigma \cap \Gamma \times [a, b]$  with  $|\lambda| \geq 2R_0$ ,

$$(5.6) \quad \sum \tilde{f}_i(\hat{\lambda})(\hat{\mu}_i^\varepsilon - \hat{\lambda}_i) \geq \min_{\hat{\zeta} \in \bar{W} \cap \Gamma \times [a, b]} \sum \tilde{f}_i(\hat{\lambda})(\hat{\zeta}_i - \lambda_i) \geq \Theta_{\bar{W}}, \quad \forall \hat{\mu} \in K_2$$

since the segment  $(\hat{\mu}^{3\varepsilon/2}, \hat{\lambda})$  must intersect  $W \cap \Gamma \times [a, b]$ . Now (5.4) follows from (5.5) and (5.6).

Finally, we note that  $\theta_K$  and  $R_K$  can be chosen so that they continuously depends on  $\sigma$ . This can be seen from the fact that the hypersurface  $\{\partial\Sigma^\sigma : \sigma \in [c, d]\}$  form a smooth foliation of the region bounded by  $\partial\Sigma^c$  and  $\partial\Sigma^d$  in  $\Gamma \times \mathbb{R}$ , which also implies that the distant function  $\text{dist}(\hat{\mu}, \partial\mathcal{B}_\sigma^+)$  also depends continuously on  $\sigma$ .  $\square$

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